THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4060 Complex Analysis
Homework 2 Suggested Solutions

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1. (Exercise 2 of Chapter 5 of [SS03]) Find the order of growth of the following entire functions:

   (a) $p(z)$ where p is a polynomial.

   (b) e^{bz^n} for $b \neq 0$.

   (c) e^{e^z}.

Solution. (a) We claim that $p(z)$ has an order of growth 0. Let $\rho > 0$. Write $p(z)$ as $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and we have that

$$\frac{|p(z)|}{e^{|z|^p}} \leq \frac{|a_n||z|^n + \cdots + |a_0|}{e^{|z|^p}} \to 0 \quad \text{as } |z| \to +\infty$$

since any exponential grows faster than any polynomial. So $|p(z)| \leq A_p e^{|z|^p}$ for some $A_p > 0$. So $p(z)$ has order of growth $\leq \rho$. Taking infimum gives $\rho_p = 0$.

(b) Let $f(z) = e^{bz^n}$ for $b \neq 0$. We show that f has order of growth n. Clearly, we have

$$|f(z)| \leq \left|e^{bz^n}\right| \leq e^{b|z|^n},$$

so $\rho_f \leq n$.

On the other hand, write $b = r_0 e^{i\theta_0}$ for some $r_0 > 0$, $0 \leq \theta_0 \leq 2\pi$ and consider $z_0 = re^{i\theta_0/n}$ for any $r > 0$. Then we see that

$$|f(z_0)| = e^{bz_0^n} = e^{r_0 e^{i\theta_0}(re^{i\theta_0/n})^n} = e^{r_0 r^n}.$$

If $|f(z)| \leq Ae^{B|z|^\rho}$, then taking $r \to +\infty$ above shows it is only possible when $\rho \geq n$, so $\rho_f = n$.

(c) We show that e^{e^z} has infinite order of growth. Let $\rho > 0$. Then

$$\lim_{x\to+\infty} \frac{e^{e^x}}{e^{Bx^\rho}} = \lim_{x\to+\infty} e^{e^x - Bx^\rho} = 0$$

so $\rho_{e^{e^z}} = +\infty$.

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2. (Exercise 7 of Chapter 5 of [SS03]) Establish the following properties of infinite products.

   (a) Show that if $\sum |a_n|^2$ converges, then the product $\prod(1 + a_n)$ converges to a non-zero limit if and only if $\sum a_n$ converges.

   (b) Find an example of a sequence of complex numbers $\{a_n\}$ such that $\sum a_n$ converges but $\prod(1 + a_n)$ diverges.

(c) Also find an example such that $\prod(1+a_n)$ converges and $\sum a_n$ diverges.

Solution. (a) Since $\sum |a_n|^2$ converges, we know that $\lim\limits_{n\to+\infty} |a_n| = 0$, so we can suppose that for n large enough, $|a_n| \leq \frac{1}{2}$. Hence, without loss of generality we assume that $|a_n| \leq \frac{1}{2}$ for all $n \in \mathbb{N}$ in the sequel.

We have that

$$\log(1+a_n) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} a_n^k = a_n + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} a_n^k = a_n + a_n^2 \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} a_n^{k-2}.$$

So we have

$$\begin{aligned}
|\log(1+a_n) - a_n| &= \left| a_n + a_n^2 \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} a_n^{k-2} - a_n \right| = \left| a_n^2 \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} a_n^{k-2} \right| \\
&\leq |a_n|^2 \sum_{k=2}^{\infty} \frac{|a_n|^{k-2}}{k} \leq |a_n|^2 \sum_{k=2}^{\infty} |a_n|^{k-2} \\
&\leq |a_n|^2 \sum_{k=2}^{\infty} \frac{1}{2^{k-2}} \leq C|a_n|^2
\end{aligned}$$

for some universal constant $C > 0$. This means that since $\sum |a_n|^2$ converges, then

$$\sum_{n=1}^{\infty} |\log(1+a_n) - a_n| \leq \sum_{n=1}^{\infty} C|a_n|^2$$

converges.

Now suppose that $\prod\limits_{n=1}^{\infty}(1+a_n) = \exp\left(\sum\limits_{n=1}^{\infty} \log(1+a_n)\right)$ converges to a non-zero limit. In particular this implies that $\sum\limits_{n=1}^{\infty} \log(1+a_n)$ converges. Then by above that

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} [a_n - \log(1+a_n) + \log(1+a_n)] \\
&= \sum_{n=1}^{\infty} \log(1+a_n) - \sum_{n=1}^{\infty} (\log(1+a_n) - a_n)
\end{aligned}$$

converges.

On the other hand, suppose that $\sum\limits_{n=1}^{\infty} a_n$ converges. Then we see that

$$\prod_{n=1}^{\infty}(1+a_n) = \exp\left(\sum_{n=1}^{\infty} \log(1+a_n)\right) = \exp\left(\sum_{n=1}^{\infty} (\log(1+a_n) - a_n) + \sum_{n=1}^{\infty} a_n\right)$$

converges to a non-zero limit since both sums on the right hand side converges.

(b) Take $a_n = \dfrac{(-1)^n}{\sqrt{n}}$. Then since $\left(\dfrac{1}{\sqrt{n}}\right)$ is a decreasing sequence and $\displaystyle\lim_{n\to+\infty} \frac{1}{\sqrt{n}} = 0$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test. On the other hand, we see that by Taylor expansion around 0,

$$\begin{aligned}\log\left(1+\frac{(-1)^n}{\sqrt{n}}\right) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{(-1)^n}{\sqrt{n}}\right)^k \\ &= \frac{(-1)^n}{\sqrt{n}} + \frac{(-1)^3(-1)^{2n}}{2n} + \sum_{k=3}^{\infty} \frac{(-1)^{k(n+1)}}{kn^{\frac{k}{2}}} \\ &= \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n} + O\left(\frac{1}{n^{3/2}}\right).\end{aligned}$$

Then we see that

$$\sum_{n=1}^{\infty} \log\left(1+\frac{(-1)^n}{\sqrt{n}}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} - \sum_{n=1}^{\infty} \frac{1}{2n} + \sum_{n=1}^{\infty} O\left(\frac{1}{n^{3/2}}\right)$$

which diverges since it contains the harmonic series. Hence,

$$\prod_{n=1}^{\infty}(1+a_n) = \exp\left(\sum_{n=1}^{\infty} \log\left(1+\frac{(-1)^n}{\sqrt{n}}\right)\right)$$

diverges.

(c) Take $a_n = -1$, then we see that $\sum_{n=1}^{\infty}(-1)$ diverges, but $\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}(1-1) = \prod_{n=1}^{\infty} 0 = 0$.

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3. (Exercise 10 of Chapter 5 of [SS03]) Find the Hadamard products for:

   (a) $e^z - 1$.

   (b) $\cos \pi z$.

   [Hint: The answers are $e^{z/2}\prod_{n=1}^{\infty}(1 + z^2/4n^2\pi^2)$ and $\prod_{n=0}^{\infty}(1 - 4z^2/(2n+1)^2))$ respectively.]

Solution. (a) Using the Hadamard product for $\sin z = z\prod_{n=1}^{\infty}\left(1 - \dfrac{z^2}{\pi^2 n^2}\right)$, we have

$$\begin{aligned}e^z - 1 = e^{z/2}\left(e^{z/2} - e^{-z/2}\right) = 2e^{z/2}\sinh(z/2) &= -i2e^{z/2}\sinh\left(\frac{iz}{2}\right) \\ = -i2e^{z/2}\left(\frac{iz}{2}\right)\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4\pi^2 n^2}\right) \\ = ze^{z/2}\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4\pi^2 n^2}\right).\end{aligned}$$

(b) Using the identity $\sin(2z) = 2\sin z\cos z$, and the fact that

$$\sin(2z) = 2z\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{\pi^2 n^2}\right),$$

we have

$$\begin{aligned}
\cos z &= \frac{1}{2}\cdot 2z\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{\pi^2 n^2}\right)\left(z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right)\right)^{-1} \\
&= \frac{\left(1-\frac{4z^2}{\pi^2}\right)\left(1-\frac{4z^2}{4\pi^2}\right)\left(1-\frac{4z^2}{9\pi^2}\right)\left(1-\frac{4z^2}{16\pi^2}\right)\left(1-\frac{4z^2}{25\pi^2}\right)\left(1-\frac{4z^2}{36\pi^2}\right)\cdots}{\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\left(1-\frac{z^2}{9\pi^2}\right)\left(1-\frac{z^2}{16\pi^2}\right)\left(1-\frac{z^2}{25\pi^2}\right)\left(1-\frac{z^2}{36\pi^2}\right)\cdots} \\
&= \prod_{n\text{ odd}}\left(1 - \frac{4z^2}{\pi^2 n^2}\right) \\
&= \prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{(2n+1)^2}\right).
\end{aligned}$$

Note that the two solutions above have the following issues: The first is that it is not clear that the resulting infinite products are indeed the infinite product that one obtains by applying the Hadamard Factorization Theorem directly. The above solution for part (b) also has the issue that the cancellation of infinitely many products needs to be justified. More correct solutions for both parts is presented below:

(a) Clearly we have that $|e^z - 1| \le e^{|z|}$ for all z, and hence $e^z - 1$ has order of growth ≤ 1. Moreover, suppose $e^z - 1$ has order of growth $\rho < 1$. Then considering $r > 0$, we have that there are some positive constants $A, B > 0$ such that

$$e^r - 1 \le Ae^{Br^\rho} \Rightarrow 1 \le \frac{Ae^{Br^\rho}}{e^r - 1}$$

but we see that since $\rho < 1$, the right hand side vanishes as $r \to +\infty$ and a contradiction arises. Hence, $e^z - 1$ has order of growth exactly 1.

The function $e^z - 1$ has simple zeros at $2k\pi i$ for $k \in \mathbb{Z}$. Then the Hadamard Factorization Theorem gives

$$\begin{aligned}
e^z - 1 &= e^{A+Bz} z\prod_{k=1}^{\infty} E_1(z/2k\pi i)E_1(-z/2k\pi i) \\
&= e^{A+Bz} z\prod_{k=1}^{\infty}\left(1 - \frac{z}{2k\pi i}\right)e^{\frac{z}{2k\pi i}}\left(1 + \frac{z}{2k\pi i}\right)e^{-\frac{z}{2k\pi i}} \\
&= e^{A+Bz} z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{4k^2\pi^2}\right).
\end{aligned}$$

It remains to determine the constants A, B. Since

$$\lim_{|z|\to 0}\frac{e^z - 1}{z} = 1$$

we have that $A = 0$. Moreover, multiplying by $e^{-z/2}$ on both sides we have

$$e^{z/2} - e^{-z/2} = e^{(B-1/2)z} z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{4k^2\pi^2}\right)$$

and observe that both $e^{z/2} - e^{-z/2}$ and $z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{4k^2\pi^2}\right)$ are odd functions, hence $e^{(B-1/2)z}$ is an even function, which forces $B = 1/2$. So we conclude that

$$e^z - 1 = e^{z/2} z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{4k^2\pi^2}\right).$$

(b) Note that for $z = x + iy$ with $x, y \in \mathbb{R}$, we have

$$|\cos \pi z| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{2}\right| \leq \frac{1}{2}(|e^{i\pi z}| + |e^{-i\pi z}|) = \frac{1}{2}(e^{-\pi y} + e^{\pi y}) \leq e^{\pi |z|}.$$

Hence, $\cos \pi z$ has order of growth ≤ 1. Moreover, the zeroes of $\cos \pi z$ are $\pm \frac{(2k-1)}{2}$ for $k = 1, 2, 3, \ldots$ and are all simple. Since $\sum_{k=1}^{\infty} 2 \cdot \frac{2}{(2k-1)}$ diverges, we see that the order of growth of $\cos \pi z \geq 1$ and hence the order of growth of $\cos \pi z$ is exactly 1.

Then by the Hadamard Factorization Theorem, there are constants A, B such that

$$\begin{aligned}
\cos \pi z &= e^{A+Bz} \prod_{k=1}^{\infty} E_1\left(\frac{z}{\frac{(2k-1)}{2}}\right) E_1\left(-\frac{z}{\frac{(2k-1)}{2}}\right) \\
&= e^{A+Bz} \prod_{k=1}^{\infty} \left(1 - \frac{2z}{(2k-1)}\right) e^{\frac{2z}{(2k-1)}} \left(1 + \frac{2z}{(2k-1)}\right) e^{-\frac{2z}{(2k-1)}} \\
&= e^{A+Bz} \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right).
\end{aligned}$$

It remains to determine the constants A, B. Since it is known that $\cos 0 = 1$, putting $z = 0$ in the equation above gives

$$1 = e^A \Rightarrow A = 0.$$

Since $\cos \pi z$ is known to be even, and clearly the infinite product is also even, we finally require that e^{Bz} is an even function. However, this forces $B = 0$ as well. Hence, we conclude that

$$\cos \pi z = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right).$$

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4. (Exercise 14 of Chapter 5 of [SS03]) Deduce from Hadamard's theorem that if F is entire and of growth order ρ that is non-integral, then F has infinitely many zeros.

Solution. Since ρ is non-integral, there is an integer k such that $k < \rho < k+1$. Suppose for the sake of contradiction that F has finitely many zeros, say $a_1, a_2, \ldots, a_N$. Then Hadamard's theorem gives

$$F(z) = e^{P(z)} z^m \prod_{n=1}^{N} E_k\left(\frac{z}{a_n}\right) = e^{P(z)} z^m \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{z^2}{2a_n} + \cdots + \frac{z^k}{k a_n}}$$

where $P(z)$ is a polynomial of degree k. Then we see that F can be written in the form $G(z)e^{Q(z)}$ where both $G(z)$ and $Q(z)$ are polynomials and in particular $Q(z)$ has degree k. So we see that F has order of growth $k < \rho$, a contradiction. ◀

References

[SS03] Elias M. Stein and Rami Shakarchi. *Complex Analysis*. Princeton University Press, 2003.